Hankel Tensors: Associated Hankel Matrices and Vandermonde Decomposition

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Outline

1. Introduction

In 1861, **H. Hankel** started the research on **Hankel matrices**. Since then, Hankel matrices play an important role in linear algebra and its applications. As a natural extension of Hankel matrices, **Hankel tensors** arise from applications such as signal processing.

Denote $[n] := \{1, \cdots, n\}$. Let $\mathcal{A} = (a_{i_1 \cdots i_m})$ be a real mth order n-dimensional tensor. If there is a vector $\mathbf{v} = (v_0, v_1, \cdots, v_{(n-1)m})^\top$ such that for $i_1, \cdots, i_m \in [n]$, we have

$$a_{i_1 \cdots i_m} \equiv v_{i_1+i_2+\cdots+i_m-m}, \tag{1}$$

then we say that $\mathcal{A}$ is an mth order **Hankel tensor**. Hankel tensors were introduced by Papy, De Lathauwer and Van Huffel in 2005 in the context of the harmonic retrieval problem, which is at the heart of many signal processing applications. In 2008, Badeau and Boyer proposed fast higher-order singular value decomposition (HOSVD) for third order Hankel tensors.

1.1. The Hankel Tensor Space

A real mth order n-dimensional tensor (hypermatrix) $\mathcal{A} = (a_{i_1 \cdots i_m})$ is a multi-array of real entries $a_{i_1 \cdots i_m}$, where $i_j \in [n]$ for $j \in [m]$. Denote the set of all real mth order n-dimensional tensors by $T_{m,n}$. Then $T_{m,n}$ is a linear space of dimension n^m. If the entries $a_{i_1 \cdots i_m}$ are invariant under any permutation of their indices, then $\mathcal{A}$ is a **symmetric tensor**. Denote the set of all real mth order n-dimensional symmetric tensors by $S_{m,n}$. Then $S_{m,n}$ is a linear subspace of $T_{m,n}$. Clearly, a Hankel tensor is a symmetric tensor. Denote the set of all real mth order n-dimensional Hankel tensors by $H_{m,n}$. Then $H_{m,n}$ is a linear subspace of $S_{m,n}$, with dimension $(n-1)m+1$.

Throughout this talk, we assume that $m, n \geq 2$. We use small letters $x, u, v, \alpha, \cdots$, for scalers, small bold letters $\mathbf{x}, \mathbf{y}, \mathbf{u}, \cdots$, for vectors, capital letters $A, B, \cdots$, for matrices, calligraphic letters $\mathcal{A}, \mathcal{B}, \cdots$, for tensors. Denote $\mathbf{e}_i \in \Re^n$ as the ith unit vector for $i \in [n]$, and $\mathbf{0}$ as the zero vector in $\Re^n$.

1.2. Motivation

People used to think that tensor (hypermatrix) problems are hard while matrix problems are tractable. This is only partially true. Actually, matrices are special cases of tensors (hypermatrices) with order two. Thus, special tensor (hypermatrix) problems may be tractable if tensors (hypermatrices) in these problems have simple structures. Since a Hankel tensor $\mathcal{A}$ is defined by a vector $\mathbf{v}$, we believe that the Hankel tensor problem is tractable. On the other hand, Hankel tensors arise from applications, and Hankel matrices have a profound theory. These three factors stimulated us to study Hankel tensors.

1.3. The Paper

This talk is based upon the following paper:

[1]. L. Qi, "Hankel tensors: Associated Hankel matrices and Vandermonde decomposition", October 2013. arXiv:1310.5470v2.

1.4. Positive Semidefinite Tensors and Copositive Tensors

Let $\mathcal{A} = (a_{i_1 \cdots i_m}) \in S_{m,n}$ and $\mathbf{x} = (x_1, \cdots, x_n)^\top \in \Re^n$. Denote

$$\mathcal{A}\mathbf{x}^m = \sum_{i_1,\cdots,i_m=1}^{n} a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}.$$

Denote $\Re^n_+ = \{\mathbf{x} \in \Re^n : \mathbf{x} \geq \mathbf{0}\}$. If $\mathcal{A}\mathbf{x}^m \geq 0$ for all $\mathbf{x} \in \Re^n_+$, then $\mathcal{A}$ is called **copositive**. If $\mathcal{A}\mathbf{x}^m > 0$ for all $\mathbf{x} \in \Re^n_+, \mathbf{x} \neq \mathbf{0}$, then $\mathcal{A}$ is called **strongly copositive**. Suppose that m is even. If $\mathcal{A}\mathbf{x}^m \geq 0$ for all $\mathbf{x} \in \Re^n$, then $\mathcal{A}$ is called **positive semi-definite**. If $\mathcal{A}\mathbf{x}^m > 0$ for all $\mathbf{x} \in \Re^n, \mathbf{x} \neq \mathbf{0}$, then $\mathcal{A}$ is called **positive definite**. Positive semi-definite symmetric tensors are useful in automatical control and higher-order diffusion tensor imaging. It is established by Qi in 2005 that an even order symmetric tensor $\mathcal{A} \in S_{m,n}$ is positive semi-definite if and only if all of its H-eigenvalues (or Z-eigenvalues) are nonnegative. On the other hand, copositive tensors do not restrict the order to be even, thus are more general. Nonnegative tensors, positive semi-definite tensors and Laplacian tensors are copositive tensors.

1.5. Copositive Hankel Tensors

We first give a necessary condition for a Hankel tensor to be copositive.

Proposition 1.1 *Suppose that $\mathcal{A} \in H_{m,n}$ is defined by (1). If $\mathcal{A}$ is copositive, then $v_{(i-1)m} \geq 0$ for $i \in [n]$.*

Proof. Since $v_{(i-1)m} = \mathcal{A}(\mathbf{e}_i)^m$ for $i \in [n]$, the conclusion follows from the definition of copositive tensors. $\square$

As a positive semi-definite symmetric tensor is copositive, the condition of Proposition 1.1 is also a necessary condition for an even order Hankel tensor to be positive semi-definite.

2. Associated Plane Tensors

For any nonnegative integer k, define $s(k,m,n)$ as the number of distinct sets of indices $(i_1,\cdots,i_m)$ such that $i_j \in [n]$ for $j \in [m]$ and $i_1+\cdots+i_m-m=k$. Then $s(0,m,n)=1$, $s(1,m,n)=m$, $s(2,m,n)=\frac{m(m+1)}{2},\cdots$.

We now definite the associated plane tensor of a Hankel tensor. Suppose that $\mathcal{A} \in H_{m,n}$ is defined by (1). Define $\mathcal{P}=(p_{i_1\cdots i_{(n-1)m}}) \in S_{(n-1)m,2}$ by

$$p_{i_1\cdots i_{(n-1)m}} = \frac{s(k,m,n)v_k}{\binom{(n-1)m}{k}},$$

where $k=i_1+\cdots+i_{(n-1)m}-(n-1)m$. We call $\mathcal{P}$ the **associated plane tensor** of $\mathcal{A}$.

2.1. Theorem 2.1

Theorem 2.1 *If a Hankel tensor $\mathcal{A} \in H_{m,n}$ is copositive, then its associated plane tensor $\mathcal{P}$ is copositive. If an even order Hankel tensor $\mathcal{A} \in H_{m,n}$ is positive semi-definite, then its associated plane tensor $\mathcal{P}$ is positive semi-definite.*

Proof. Suppose that $\mathcal{A}$ is copositive. By Proposition 1.1, $v_{(n-1)m} \geq 0$. Let $\mathbf{y} = (y_1, y_2)^\top \in \Re^2_+$. If $y_1 = y_2 = 0$, then clearly $\mathcal{P}\mathbf{y}^{(n-1)m} = 0$. If $y_1 = 0$ and $y_2 \neq 0$, then $\mathcal{P}\mathbf{y}^{(n-1)m} = v_{(n-1)m} y_2^{(n-1)m} \geq 0$. We now assume that $y_1 \neq 0$. Let $u = \frac{y_2}{y_1}$. Then $u \geq 0$. We have

$$\mathcal{P}\mathbf{y}^{(n-1)m} = y_1^{(n-1)m} \sum_{k=0}^{(n-1)m} \binom{(n-1)m}{k} \cdot \frac{s(k,m,n)v_k}{\binom{(n-1)m}{k}} u^k = y_1^{(n-1)m} \mathcal{A}\mathbf{u}^m \geq 0, \tag{2}$$

where $\mathbf{u} = (1, u, u^2, \cdots, u^{n-1})^\top \in \Re^n_+$. Thus, $\mathcal{P}$ is copositive.

Suppose that m is even and $\mathcal{A}$ is positive semi-definite. Then $(n-1)m$ is also even. By Proposition 1.1, $v_{(n-1)m} \geq 0$. Let $\mathbf{y} = (y_1, y_2)^\top \in \Re^2$. If $y_1 = y_2 = 0$, then clearly $\mathcal{P}\mathbf{y}^{(n-1)m} = 0$. If $y_1 = 0$ and $y_2 \neq 0$, then $\mathcal{P}\mathbf{y}^{(n-1)m} = v_{(n-1)m} y_2^{(n-1)m} \geq 0$. We now assume that $y_1 \neq 0$. Let $u = \frac{y_2}{y_1}$. Then $u \neq 0$. The derivation (2) still holds with $\mathbf{u} = (1, u, u^2, \cdots, u^{n-1})^\top \in \Re^n$. Thus, $\mathcal{P}$ is positive semi-definite. $\square$

2.2. Questions

We may use the methods in Qi, Wang and Wang (2009) to check if $\mathcal{P}$ is positive semi-definite or not when m is even. In [1], we presented an algorithm for checking if $\mathcal{P}$ is copositive or not.

Can we give an example that $\mathcal{P}$ is copositive but $\mathcal{A}$ is not? When m is even, can we give an example that $\mathcal{P}$ is positive semi-definite but $\mathcal{A}$ is not? Which conditions on $\mathcal{P}$ may assure co-positiveness or positive semi-definiteness of $\mathcal{A}$?

3. Generating Functions and Strong Hankel Tensors

Suppose that $\mathcal{A} \in H_{m,n}$ is defined by (1). Let $A = (a_{ij})$ be an $\lceil \frac{(n-1)m+2}{2} \rceil \times \lceil \frac{(n-1)m+2}{2} \rceil$ matrix with $a_{ij} \equiv v_{i+j-2}$, where $v_{2\lceil \frac{(n-1)m}{2} \rceil}$ is an additional number when $(n-1)m$ is odd. Then A is a Hankel matrix, associated with the Hankel tensor $\mathcal{A}$. Such an associated Hankel matrix is unique if $(n-1)m$ is even. If the Hankel matrix A is positive semi-definite, then we say that $\mathcal{A}$ is a **strong Hankel tensor**.

3.1. A Generating Function

Let $\mathcal{A}$ be a Hankel tensor defined by (1). Let $f(t)$ be an absolutely integrable real valued function on the real line $(-\infty, \infty)$ such that

$$v_k \equiv \int_{-\infty}^{\infty} t^k f(t) dt, \tag{3}$$

for $k = 0, \cdots, (n-1)m$. Then we say that f is a **generating function** of the Hankel tensor $\mathcal{A}$. We see that $f(t)$ is also the generating function of the associated Hankel matrix of $\mathcal{A}$. By the theory of Hankel matrices, $f(t)$ is well-defined.

3.2. Theorem 3.1

Theorem 3.1 *A Hankel tensor $\mathcal{A}$ has a nonnegative generating function if and only if it is a strong Hankel tensor. An even order strong Hankel tensor is positive semi-definite.*

On the other hand, suppose that $\mathcal{A} \in H_{m,n}$ has a generating function $f(t)$ such that (3) holds. If $\mathcal{A}$ is copositive, then

$$\int_{-\infty}^{\infty} t^{(i-1)m} f(t) dt \geq 0$$

for $i \in [n]$.

3.3. Proof

Proof. By the famous Hamburger moment problem, such a nonnegative generating function exists if and only if the associated Hankel matrix is positive semi-definite, i.e., $\mathcal{A}$ is a strong Hankel tensor. On the other hand, suppose that $\mathcal{A}$ has such a nonnegative generating function f and m is even. Then for any $\mathbf{x} \in \Re^n$, we have

$$\begin{aligned} \mathcal{A}\mathbf{x}^m &= \sum_{i_1,\cdots,i_m=1}^{n} a_{i_1\cdots i_m} x_{i_1} \cdots x_{i_m} \\ &= \sum_{i_1,\cdots,i_m=1}^{n} \int_{-\infty}^{\infty} t^{i_1+\cdots+i_m-m} x_{i_1} \cdots x_{i_m} f(t) dt \\ &= \int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} x_i t^{i-1} \right)^m f(t) dt \\ &\geq 0. \end{aligned}$$

Thus, if m is even and $\mathcal{A}$ is a strong Hankel tensor, then $\mathcal{A}$ is positive semi-definite.

The final conclusion follows from (3) and Proposition 1.1. □

3.4. A Counter Example

We now give an example of a positive semi-definite Hankel tensor, which is not a strong Hankel tensor. Let $m = 4$ and $n = 2$. Let $v_0 = v_4 = 1$, $v_2 = -\frac{1}{6}$, and $v_1 = v_3 = 0$. Let $\mathcal{A}$ be defined by (1). Then for any $\mathbf{x} \in \Re^2$, we have

$$\mathcal{A}\mathbf{x}^4 = v_0 x_1^4 + 4v_1 x_1^3 v_2 + 6v_2 x_1^2 x_2^2 + 4v_3 x_1 x_2^3 + v_4 x_2^4 = x_1^4 - x_1^2 x_2^2 + x_2^4 \geq 0.$$

Thus, $\mathcal{A}$ is positive semi-definite. Let A be the unique Hankel matrix associated with $\mathcal{A}$. Since $v_2 < 0$, by Proposition 1.1, A is not positive semi-definite. Thus, $\mathcal{A}$ is not a strong Hankel tensor.

The question is, for a fixed even number $m \geq 4$, can we characterize a positive semi-definite Hankel tensor by its generating functions? If the associated Hankel matrix is copositive, is the Hankel tensor copositive?

3.5. Hadamard Product

We now discuss the Hadamard product of two strong Hankel tensors. Let $\mathcal{A} = (a_{i_1\cdots i_m}), \mathcal{B} = (b_{i_1\cdots i_m}) \in T_{m,n}$. Define the Hadamard product of $\mathcal{A}$ and $\mathcal{B}$ as $\mathcal{A} \circ \mathcal{B} = (a_{i_1\cdots i_m} b_{i_1\cdots i_m}) \in T_{m,n}$. Clearly, the Hadamard product of two Hankel tensors is a Hankel tensor.

Proposition 3.1 *The Hadamard product of two strong Hankel tensors is a strong Hankel tensor.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two strong Hankel tensors in $H_{m,n}$. Let A and B be Hankel matrices associated with $\mathcal{A}$ and $\mathcal{B}$ respectively, such that A and B are positive semi-definite. Clearly, the Hadamard product of A and B is a Hankel matrix associated with the Hadamard product of $\mathcal{A}$ and $\mathcal{B}$. By the Shur product theorem, the Hadamard product of two positive semi-definite symmetric matrices is still a positive semi-definite symmetric matrix. Thus, the Hadamard product of A and B is positive semi-definite. This implies that the Hadamard product of $\mathcal{A}$ and $\mathcal{B}$ is a strong Hankel tensor. $\square$

3.6. A Counter Example

On the other hand, the Hadamard product of two positive semi-definite Hankel tensors may not be positive semi-definite. Assume that $m = 4$ and $n = 2$. Let $\mathcal{A}$ be the example given above. Then $\mathcal{A}$ is a positive semi-definite Hankel tensor. On the other hand, let $\mathcal{B} = (b_{i_1i_2i_3i_4}) \in S_{4,2}$ be defined by $b_{i_1i_2i_3i_4} = 1$ if $i_1 + i_2 + i_3 + i_4 = 6$, and $b_{i_1i_2i_3i_4} = 0$ otherwise. We may verify that $\mathcal{B}$ is a strong Hankel tensor, thus a positive semi-definite Hankel tensor. It is easy to verify that $\mathcal{A} \circ \mathcal{B}$ is not positive semi-definite. Note here that $\mathcal{A}$ is not a strong Hankel tensor. Thus, this example does not contradict Proposition 3.1.

4. Vandermonde Decomposition and Complete Hankel Tensors

For any vector $\mathbf{u} \in \Re^n$, $\mathbf{u}^m$ is a rank-one mth order symmetric n-dimensional tensor $\mathbf{u}^m = (u_{i_1} \cdots u_{i_m}) \in S_{m,n}$. If $\mathbf{u} = (1, u, u^2, \cdots, u^{n-1})^\top$, then $\mathbf{u}$ is called a **Vandermonde vector**. If

$$\mathcal{A} = \sum_{k=1}^{r} \alpha_k \left(\mathbf{u}_k\right)^m, \tag{4}$$

where $\alpha_k \in \Re$, $\alpha_k \neq 0$, $\mathbf{u}_k = (1, u_k, u_k^2, \cdots, u_k^{n-1})^\top \in \Re^n$ are Vandermonde vectors for $k = 1, \cdots, r$, and $u_i \neq u_j$ for $i \neq j$, then we say that tensor $\mathcal{A}$ has a **Vandermonde decomposition**. We call the minimum value of r the **Vandermonde rank** of $\mathcal{A}$.

4.1. Theorem 4.1

Theorem 4.1 *Let $\mathcal{A} \in S_{m,n}$. Then $\mathcal{A}$ is a Hankel tensor if and only if it has a Vandermonde decomposition (4). In this case, we have $r \leq (n-1)m+1$. Suppose that $\mathcal{A}$ has a Vandermonde decomposition (4). If $\mathcal{A}$ is copositive, then*

$$\sum_{k=1}^{r} \alpha_k u_k^{(i-1)n} \geq 0, \quad \text{for } i \in [n]. \tag{5}$$

On the other other hand, if m is even and $\alpha_k > 0$ for $i \in [r]$, then $\mathcal{A}$ is positive semi-definite.

4.2. Proof

Proof. Suppose that $\mathcal{A}$ has a Vandermonde decomposition (4). Let

$$v_i = \sum_{k=1}^{r} \alpha_k u_k^i, \quad \text{for } i = 0, \cdots, (m-1)n. \tag{6}$$

By (4), we see that (1) holds. Thus, $\mathcal{A}$ is a Hankel tensor.
On the other hand, assume that $\mathcal{A}$ is a Hankel tensor defined by (1). Let $r = (m-1)n + 1$. Pick real numbers $u_k, k \in [r]$ such that $u_i \neq u_j$ for $i \neq j$. By matrix analysis, the coefficient matrix of the linear system (6) with $\alpha_k, k \in [r]$ as variables, is a Vandermonde matrix, which is nonsingular. Thus, the linear system (6) has a solution $\alpha_k, k \in [r]$. Substituting such $\alpha_k, k = 1, \cdots, r$ to (4), we see that (4) holds, i.e., $\mathcal{A}$ has a Vandermonde decomposition.
Suppose that $\mathcal{A}$ has a Vandermonde decomposition (4). If $\mathcal{A}$ is copositive, then (5) follows from (6) and Proposition 1.1. On the other hand, assume that m is even. Suppose (4) holds with $\alpha_k > 0, k \in [r]$. For any $\mathbf{x} \in \Re^n$, we have

$$\mathcal{A}\mathbf{x}^m = \sum_{k=1}^{r} \alpha_k (\mathbf{u}_k^\top \mathbf{x})^m \geq 0.$$

Thus, $\mathcal{A}$ is positive semi-definite. $\square$

4.3. Complete Hankel Tensors

In (4), if $\alpha_k > 0, k \in [r]$, then we say that $\mathcal{A}$ has a positive Vandermonde decomposition and call $\mathcal{A}$ a **complete Hankel Tensor**. Thus, Theorem 4.1 says that an even order complete Hankel tensor is positive semi-definite. We will study the spectral properties of odd order complete Hankel tensors in the next section.

By (6), if $\alpha_k > 0$ for $k \in [r]$, then v_i is nonnegative if i is even. Thus, the counterexample $\mathcal{A}$, given in the last section, is not a complete Hankel tensor as it has $v_2 < 0$. This implies that a positive semi-definite Hankel tensor may not be a complete Hankel tensor.

4.4. Hadamard Product

Proposition 4.1 *The Hadamard product of two complete Hankel tensors is a complete Hankel tensor.*

Proof. Suppose that $\mathcal{A}, \mathcal{B} \in H_{m,n}$ are two complete Hankel tensors. Then we may assume that each of $\mathcal{A}$ and $\mathcal{B}$ has a positive Vandermonde decomposition:

$$\mathcal{A} = \sum_{k=1}^{r} \alpha_k \left(\mathbf{u}_k\right)^m$$

and

$$\mathcal{B} = \sum_{j=1}^{s} \beta_j \left(\mathbf{v}_j\right)^m,$$

where $\alpha_k > 0, \mathbf{u}_k = (1, u_k, u_k^2, \cdots, u_k^{n-1})^\top$ are Vandermonde vectors for $k \in [r]$, $\beta_j > 0, \mathbf{v}_j = (1, v_j, v_j^2, \cdots, v_j^{n-1})^\top$ are Vandermonde vectors for $j \in [s]$. Then the Vandermonde product of $\mathcal{A}$ and $\mathcal{B}$ is

$$\mathcal{A} \circ \mathcal{B} = \sum_{k=1}^{r} \sum_{j=1}^{s} \alpha_k \beta_j \left(\mathbf{w}_{kj}\right)^\top,$$

where $\alpha_k \beta_j > 0, \mathbf{w}_{kj} = (1, u_k v_j, (u_k v_j)^2, \cdots, (u_k v_j)^{n-1})^\top$ are Vandermonde vectors for $k \in [r]$ and $j \in [s]$. We see that $\mathcal{A} \circ \mathcal{B}$ has a positive Vandermonde decomposition, thus a complete Hankel tensor. $\square$

4.5. Summary

We may summarize the results on Hadamard products. The Hadarmard product of two Hankel tensors is a Hankel tensor. The Hadarmard product of two strong Hankel tensors is a strong Hankel tensor. The Hadarmard product of two complete Hankel tensors is a complete Hankel tensor. But the Hadarmard product of two positive semi-definite Hankel tensors may not be positive semi-definite.

Can we characterize a positive semi-definite Hankel tensor by its Vandermonde decomposition? Is a strong Hankel tensor a complete Hankel tensor? Is a complete Hankel tensor a strong Hankel tensor?

5. Spectral Properties of Odd Order Hankel Tensors

Suppose that m is even. Then by Theorem 5 of Qi (2005), all the H-eigenvalues and Z-eigenvalues of a strong Hankel tensor or a complete Hankel tensor are nonnegative, as strong Hankel tensors and complete Hankel tensors are positive semi-definite. In this section, we discuss spectral properties of odd order complete and strong Hankel tensors. Hence, assume that m is odd in this section.

5.1. Eigenvalues and Eigenvectors

We now briefly review the definition of eigenvalues, H-eigenvalues E-eigenvalues and Z-eigenvalues of a real mth order n-dimensional symmetric tensor $\mathcal{A} = (a_{i_1 \cdots i_m}) \in S_{m,n}$. Let $\mathbf{x} = (x_1, \cdots, x_n)^\top \in C^n$. Then $\mathcal{A}\mathbf{x}^{m-1}$ is an n-dimensional vector, with its ith component as $\sum_{i_2 \cdots i_m = 1}^{n} a_{ii_2 \cdots i_m} x_{i_2} \cdots x_{i_m}$. For any vector $\mathbf{x} \in C^n$, $\mathbf{x}^{[m-1]}$ is a vector in C^n, with its ith component as x_i^{m-1}. If $\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}$ for some $\lambda \in C$ and $\mathbf{x} \in C^n \setminus \{0\}$, then λ is called an **eigenvalue** of $\mathcal{A}$ and $\mathbf{x}$ is called an **eigenvector** of $\mathcal{A}$, associated with λ. If both λ and $\mathbf{x}$ are real, then they are called an **H-eigenvalue** and an **H-eigenvector** of $\mathcal{A}$, respectively. If $\mathcal{A}x^{m-1} = \lambda \mathbf{x}$ for some $\lambda \in C$ and $\mathbf{x} \in C^n$, satisfying $\mathbf{x}^\top \mathbf{x} = 1$, then λ is called an **E-eigenvalue** of $\mathcal{A}$ and $\mathbf{x}$ is called an **E-eigenvector** of $\mathcal{A}$, associated with λ. If both λ and $\mathbf{x}$ are real, then they are called a **Z-eigenvalue** and a **Z-eigenvector** of $\mathcal{A}$, respectively. Note that Z-eigenvalues always exist, and when m is even, H-eigenvalues always exist.

5.2. H-Eigenvalues of Complete Hankel Tensors

Proposition 5.1 *Suppose that m is odd and $\mathcal{A} \in H_{m,n}$ is a complete Hankel tensor. Assume that $\mathcal{A}$ has at least one H-eigenvalue. Then all the H-eigenvalues of $\mathcal{A}$ are nonnegative. Let λ be an H-eigenvalue of $\mathcal{A}$, with an H-eigenvector $\mathbf{x} = (x_1, \cdots, x_n)^\top$. Then either $\lambda = 0$ or $\lambda > 0$ with $x_1 \neq 0$.*

Proof. By the definition of complete Hankel tensors, $\mathcal{A}$ has a Vandermonde decomposition (4), with $\alpha_k > 0$ for $k \in [r]$. Suppose that $\mathcal{A}$ has an H-eigenvalue λ associated with an H-eigenvector $\mathbf{x} = (x_1, \cdots, x_n)^\top$. Then for $i \in [n]$, we have

$$\lambda x_i^{m-1} = \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i = \sum_{k=1}^{r} \alpha_k u_k^{i-1} \left[(\mathbf{u}_k)^\top \mathbf{x}\right]^{m-1}. \tag{7}$$

If $(\mathbf{u}_k)^\top \mathbf{x} = 0$ for all $k \in [r]$, then the right hand side of (7) is 0. Since $\mathbf{x} \neq \mathbf{0}$, we may pick i such that $x_i \neq 0$. Then (7) implies that $\lambda = 0$.
Suppose that $(\mathbf{u}_k)^\top \mathbf{x} \neq 0$ for at least one k. Let $i = 1$. Then the the right hand side of (7) is positive. This implies that $\lambda > 0$ and $x_1 \neq 0$. $\square$
In general an odd order symmetric tensor may not have H-eigenvalues. Does a complete Hankel tensor always have an H-eigenvalue?

5.3. Z-Eigenvalues of Complete Hankel Tensors

Proposition 5.2 *Suppose that* m *is odd and* $\mathbf{x} = (x_1, \cdots, x_n)^\top$ *is a Z-eigenvector of a complete Hankel tensor* $\mathcal{A} \in H_{m,n}$, *associated with a Z-eigenvalue* λ. *Then* $x_i \geq 0$ *for all odd* i *and* $x_1 > 0$ *if* $\lambda > 0$; *and* $x_i \leq 0$ *for all odd* i *and* $x_1 < 0$ *if* $\lambda < 0$.

Proof. Again, by the definition of complete Hankel tensors, $\mathcal{A}$ has a Vandermonde decomposition (4), with $\alpha_k > 0$ for $k \in [r]$. Suppose that $\mathcal{A}$ has a Z-eigenvalue λ associated with a Z-eigenvector $\mathbf{x} = (x_1, \cdots, x_n)^\top$. Then for $i \in [n]$, we have

$$\lambda x_i = \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i = \sum_{k=1}^{r} \alpha_k u_k^{i-1} \left[(\mathbf{u}_k)^\top \mathbf{x}\right]^{m-1}. \tag{8}$$

If $(\mathbf{u}_k)^\top \mathbf{x} = 0$ for all $k \in [r]$, then the right hand side of (8) is 0. Since $\mathbf{x} \neq \mathbf{0}$, we may pick i such that $x_i \neq 0$. Then (8) implies that $\lambda = 0$.
Suppose that $(\mathbf{u}_k)^\top \mathbf{x} \neq 0$ for at least one k. Let i be odd. Then the the right hand side of (8) is nonnegative. This implies that $\lambda x_i \geq 0$. The conclusion on x_i with i odd follows. Let $i = 1$. Then the the right hand side of (8) is positive. This implies that $\lambda x_1 > 0$. The conclusion on x_1 follows now. $\square$

5.4. Z-Eigenvalues of Strong Hankel Tensors

Proposition 5.3 *Suppose that m is odd and $\mathbf{x} = (x_1, \cdots, x_n)^\top$ is a Z-eigenvector of a strong Hankel tensor $\mathcal{A} \in H_{m,n}$, associated with a Z-eigenvalue λ. Then $x_i \geq 0$ for all odd i if $\lambda > 0$; and $x_i \leq 0$ for all odd i if $\lambda < 0$.*

Proof. By Theorem 1, $\mathcal{A}$ has a nonnegative generating function $f(t)$ such that (3) holds. Suppose that $\mathcal{A}$ has a Z-eigenvalue λ associated with a Z-eigenvector $\mathbf{x} = (x_1, \cdots, x_n)^\top$. Then for $i \in [n]$, we have

$$\begin{aligned} \lambda x_i &= \left(\mathcal{A}\mathbf{x}^{m-1}\right)_i \\ &= \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \\ &= \sum_{i_2,\cdots,i_m=1}^{n} \int_{-\infty}^{\infty} t^{i+i_2+\cdots+i_m-m} x_{i_1} \cdots x_{i_m} f(t)dt \\ &= \int_{-\infty}^{\infty} t^{i-1} \left(\sum_{i=1}^{n} x_i t^{i-1}\right)^{m-1} f(t)dt. \end{aligned} \tag{9}$$

Let i be odd. Then the the right hand side of (9) is nonnegative. The conclusion follows now. □

5.5. Odd Order Positive Semi-Definite Tensors

Note that we miss a result of the H-eigenvalues of an odd order strong Hankel tensor. Are all the H-eigenvalues of an odd order strong Hankel tensor nonnegative?

Similar spectral properties hold for odd order Laplacian tensors and odd order completely positive tensors. A common point is that such classes of symmetric tensors are positive semi-definite when the order is even. Thus, we may think if we may define some odd order "positive semi-definite" symmetric tensors, with such spectral properties. Further study is needed on such a phenomenon.

6. Bounds for the Largest and the Smallest Z-Eigenvalues

Let $\mathcal{A} \in S_{m,n}$. Then $\mathcal{A}$ always has Z-eigenvalues. Denote the smallest and the largest Z-eigenvalue of $\mathcal{A}$ by $\lambda_{\min}(\mathcal{A})$ and $\lambda_{\max}(\mathcal{A})$ respectively. We always have

$$\lambda_{\min}(\mathcal{A}) = \min\{\mathcal{A}\mathbf{x}^m : \mathbf{x} \in \Re^n, \mathbf{x}^\top \mathbf{x} = 1\} \tag{10}$$

and

$$\lambda_{\max}(\mathcal{A}) = \max\{\mathcal{A}\mathbf{x}^m : \mathbf{x} \in \Re^n, \mathbf{x}^\top \mathbf{x} = 1\}. \tag{11}$$

If m is even, $\mathcal{A}$ is positive semi-definite if and only if $\lambda_{\min}(\mathcal{A}) \geq 0$. If m is odd, then $\lambda_{\max}(\mathcal{A}) \geq 0$ and $\lambda_{\min}(\mathcal{A}) = -\lambda_{\max}(\mathcal{A})$. In general, $\max\{|\lambda_{\min}(\mathcal{A})|, |\lambda_{\max}(\mathcal{A})|\}$ is a norm of $\mathcal{A}$ in the space $S_{m,n}$. If $|\lambda_{\min}(\mathcal{A})| = \max\{|\lambda_{\min}(\mathcal{A})|, |\lambda_{\max}(\mathcal{A})|\}$, then $\lambda_{\min}(\mathcal{A})$ and its corresponding eigenvector $\mathbf{x}$ form the best rank-one approximation to $\mathcal{A}$. Similarly, if $|\lambda_{\max}(\mathcal{A})| = \max\{|\lambda_{\min}(\mathcal{A})|, |\lambda_{\max}(\mathcal{A})|\}$, then $\lambda_{\max}(\mathcal{A})$ and its corresponding eigenvector $\mathbf{x}$ form the best rank-one approximation to $\mathcal{A}$. Let $\mathbf{x} \in \Re^n, \mathbf{x} \neq \mathbf{0}$. By (10) and (11), we have

$$\lambda_{\min}(\mathcal{A}) \leq \frac{\mathcal{A}\mathbf{x}^m}{\|\mathbf{x}\|_2^m} \leq \lambda_{\max}(\mathcal{A}). \tag{12}$$

6.1. Bounds

Proposition 6.1 *Suppose that $\mathcal{A} \in H_{m,n}$. Then*

$$\lambda_{\min}(\mathcal{A}) \le \min_{i\in[n]} v_{(i-1)m} \le \max_{i\in[n]} v_{(i-1)m} \le \lambda_{\max}(\mathcal{A}).$$

Proof. Since $v_{(i-1)m} = \mathcal{A}(\mathbf{e}_i)^m$ for $i \in [n]$, the conclusion follows from (12). $\square$.

Suppose $\mathcal{P}$ is the associated plane tensor of $\mathcal{A}$. We now use $\lambda_{\min}(\mathcal{P})$ and $\lambda_{\max}(\mathcal{P})$ to give an upper bound for $\lambda_{\min}(\mathcal{A})$, and a lower bound for $\lambda_{\max}(\mathcal{A})$, respectively.

Proposition 6.2 *Suppose that $\mathcal{A} \in H_{m,n}$, and $\mathcal{P}$ is the associated plane tensor of $\mathcal{A}$. Assume that $m(n-1)$ is even. If $\mathbf{y} = (y_1, y_2)^\top$ is a Z-eigenvector of $\mathcal{P}$, associated with $\lambda_{\min}(\mathcal{P})$*

$$\sqrt{\sum_{j=0}^{(n-1)m} y_1^{2(n-1)m-2j} y_2^{2j}}\,\lambda_{\min}(\mathcal{A}) \le \lambda_{\min}(\mathcal{P}). \tag{13}$$

If $\mathbf{z} = (z_1, z_2)^\top$ is a Z-eigenvector of $\mathcal{P}$, associated with $\lambda_{\max}(\mathcal{P})$

$$\sqrt{\sum_{j=0}^{(n-1)m} z_1^{2(n-1)m-2j} z_2^{2j}}\,\lambda_{\max}(\mathcal{A}) \ge \lambda_{\max}(\mathcal{P}). \tag{14}$$

6.2. Proof

Proof. If $y_1 = 0$, since $y_1^2 + y_2^2 = 1$, then

$$\sqrt{\sum_{j=0}^{(n-1)m} y_1^{2(n-1)m-2j} y_2^{2j}} = 1.$$

We have

$$\lambda_{\min}(\mathcal{P}) = \mathcal{P}\mathbf{y}^{(n-1)m} = v_{(n-1)m} \geq \lambda_{\min}(\mathcal{A}) = \sqrt{\sum_{j=0}^{(n-1)m} y_1^{2(n-1)m-2j} y_2^{2j}} \lambda_{\min}(\mathcal{A}),$$

where the inequality is due to Proposition 6.1. Thus, (13) holds.

Suppose that $y_1 \neq 0$. Let $u = \frac{y_2}{y_1}$ and $\mathbf{u} = (1, u, u^2, \cdots, u^{n-1})^\top \in \Re^n$. Then

$$\begin{aligned}
\lambda_{\min}(\mathcal{P}) &= \mathcal{P}\mathbf{y}^{(n-1)m} \\
&= y_1^{(n-1)m} \sum_{k=0}^{(n-1)m} \binom{(n-1)m}{k} \cdot \frac{s_{k,m} v_k}{\binom{(n-1)m}{k}} u^k \\
&= \left| y_1^{(n-1)m} \right| \mathcal{A}\mathbf{u}^m \\
&= \left| y_1^{(n-1)m} \right| \|\mathbf{u}\|_2^m \frac{\mathcal{A}\mathbf{u}^m}{\|\mathbf{u}\|_2^m} \\
&= \sqrt{\sum_{j=0}^{(n-1)m} y_1^{2(n-1)m-2j} y_2^{2j}} \frac{\mathcal{A}\mathbf{u}^m}{\|\mathbf{u}\|_2^m} \\
&\geq \sqrt{\sum_{j=0}^{(n-1)m} y_1^{2(n-1)m-2j} y_2^{2j}} \lambda_{\min}(\mathcal{A}),
\end{aligned}$$

where the inequality is due to (12). Thus, (13) also holds in this case. This proves (13).

We may prove (14) similarly. □

6.3. Question

Suppose that a Hankel tensor $\mathcal{A}$ is associated with a Hankel matrix A. Can we use the largest and the smallest eigenvalues of A to bound the largest and the smallest H-eigenvalues (Z-eigenvalues) of $\mathcal{A}$?

7. Final Remarks

In this talk, we make an initial study on Hankel tensors. We see that Hankel tensors have a very special structure, hence have very special properties. We associate a Hankel tensor with a Hankel matrix, a symmetric plane tensor, generating functions and Vandermonde decompositions. They will be useful tools for further study on Hankel tensors.

Some questions have already been raised. Here are some further questions.

1. Badeau and Boyer (2008) proposed fast higher-order singular value decomposition (HOSVD) for third order Hankel tensors. Can we construct some efficient algorithms for the largest and the smallest H-eigenvalues (Z-eigenvalues) of a Hankel tensor, or a strong Hankel tensor, or a complete Hankel tensor?

2. In general, it is NP-hard to compute the largest and the smallest H-eigenvalues (Z-eigenvalues) of a symmetric tensor. What is the complexity for computing the smallest H-eigenvalues (Z-eigenvalues) of a Hankel tensor, a strong Hankel tensor, and a complete Hankel tensor?

7.1. More Questions

3. Proposition 8 of Qi (2005) says that the determinants of all the principal symmetric sub-tensors of a positive semi-definite tensor are nonnegative. The converse is not true in general. Is the converse of Proposition 8 of Qi (2005) true for Hankel tensors?

4. The theory of Hankel matrices is based upon finite and infinite Hankel matrices as well as Hankel operators. Should we also study infinite Hankel tensors and multi-linear Hankel operators?